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# The asymmetry of stress in granular media

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**Abstract**

Here, we show that the average stress in granular media, which is defined from virtual work, may be asymmetric in the absence of contact moments. We specify the circumstances and amplitude of stress asymmetry, and calculate the corresponding couple stress and first stress moment. We also show that the average stress is always symmetric, when it is alternately defined by using statics and no contact moment. The stress asymmetry, which results from external moments, has an amplitude that decreases with the volume size. The present analysis applies to two- and three-dimensional particles of arbitrary shapes. The asymmetric stress, couple stress and first stress moment are analytically calculated in a particular example with cylindrical and spherical particles. 

*Keywords:* Granular mechanics; Couple stress; Stress asymmetry; Discrete particles; Virtual work

## 1. Introduction

The definition of stress in granular media is a controversial topic in mechanics. Some researchers (Bogdanova-Bontcheva and Lippmann, 1975; Chang and Ma, 1991; Kanatani, 1979; Mühlhaus and Vardoulakis, 1987) claim that stress tensor is not symmetric in granular media, and that couple stresses are important to understand material instability such as shear banding. Others (Christoffersen et al., 1981; Cundall and Strack, 1978–1979) affirm that the stress asymmetry is absent or negligible for all practical purposes, and unnecessarily complicates the description of the mechanical behavior of granular media.

The controversy about the asymmetry of stress and the existence of couple stress is not specific to granular media. Couple stresses were proposed in metals and fracture mechanics to regularize the stress intensity at crack tips (Sternberg, 1968), but there is not yet a convincing experimental evidence for couple stress (Diepolder et al., 1991).

The objective of this article is to re-examine the definition of stress in granular materials, and to establish the conditions under which there may be couple stresses and asymmetric stress. Following the introduction,

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Sections 2 and 3 review the basic equations of granular and equivalent continuous media. Section 3 re-examines the definition of stress from virtual work and statics. Finally, Section 6 gives an example illustrating the stress asymmetry in granular media.

## 2. Granular medium

### 2.1. Definition

As shown in Fig. 1, volume $V$ is filled with $N$ particles, some of which are subjected to external forces or moments applied from the exterior of volume $V$. The particles are grouped in set $B = \{1, \dots, N\}$. The forces and moment acting on the particles of $B$ are concentrated at $M$ points of set $C = \{1, \dots, M\}$. As shown in Fig. 1, subset $I$ represents the contact points between two particles of $B$, whereas the $E$ denotes the points, where external actions are applied:

$$I = \{1, \dots, M_I\}, \quad E = \{M_{I+1}, \dots, M\} \quad \text{and} \quad C = I \cup E = \{1, \dots, M\}. \tag{1}$$

Sets $I_a$, $E_a$, and $C_a$ denote the contact points on particle $a$ corresponding to internal actions, external actions, and all actions, respectively. Sets $C_a$, $I_a$, $E_a$, $I$, $E$, and $C$ are related as follows:

$$C = \bigcup_{a \in B} C_a, \quad C_a = I_a \cup E_a, \quad I = \bigcup_{a \in B} I_a, \quad \text{and} \quad E = \bigcup_{a \in B} E_a. \tag{2}$$

The intersections of $I_a$ and $E_a$ for two different particles are either empty or reduced to a single point $c$:

$$E_a \cap E_b = \emptyset \quad \text{and} \quad I_a \cap I_b = \{c\} \qquad \forall a \neq b \in B. \tag{3}$$

The particle assembly is in equilibrium when each particle is in equilibrium. The equilibrium of internal and external forces acting on particle $a$ is

$$\sum_{c \in C_a} f_i^{ac} = 0, \quad i = 1,\ 2,\ 3, \tag{4}$$

where $f_i^{ac}$ is the force at contact $c$. The equilibrium of moments about the center of particle $a$ is

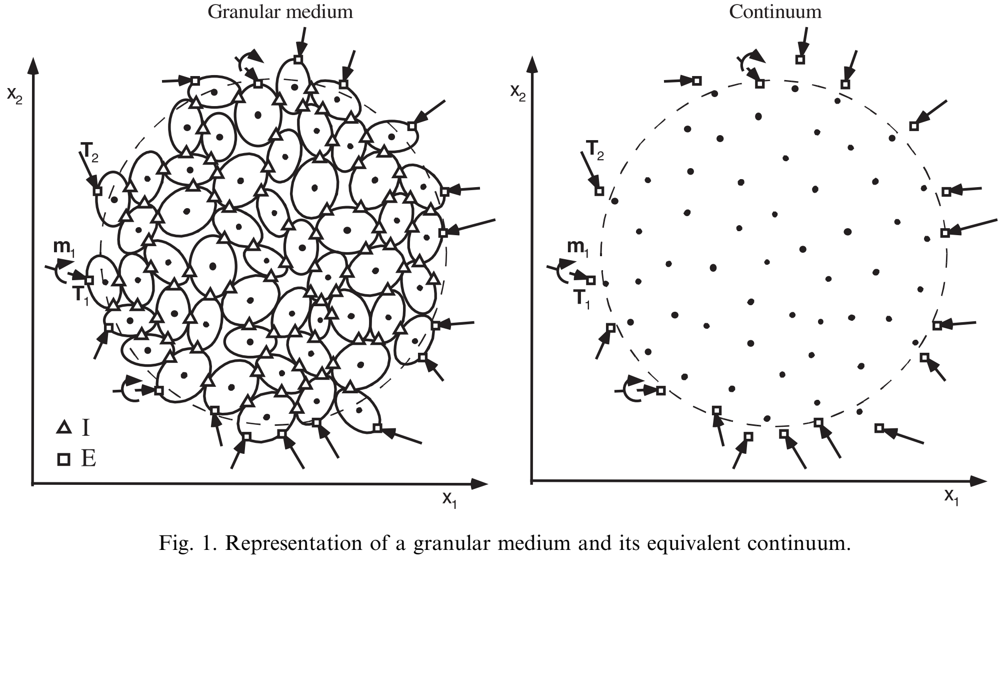


Fig. 1. Representation of a granular medium and its equivalent continuum.

$$\sum_{c \in C_a} \left( m_i^{ac} + e_{ijk}(x_j^c - x_j^a) f_k^{ac} \right) = 0, \quad i = 1,\ 2,\ 3, \tag{5}$$

where $m_i^{ac}$ represents the internal or external moments at contact point $c$, $e_{ijk}$ is the permutation symbol used for vector cross-product, and $x_j^a$ and $x_j^c$ are the coordinates of particle center $a$ and contact point $c$, respectively.

### 2.2. Virtual work in granular media

As shown in Fig. 1, the kinematics of granular media is represented by the displacement $\delta u_i^a$ of the particle centers, and the particle rotation $\delta\theta_i^a$. After multiplying Eqs. (4) and (5) by any virtual displacement $\delta u_i^a$ and rotation $\delta\theta_i^a$, and summing for all the particles of volume $V$, one obtains the following relation:

$$\sum_{a \in B} \sum_{c \in C_a} \left( f_i^{ac}\, \delta u_i^a + \left( m_i^{ac} + e_{ijk}(x_j^c - x_j^a) f_k^{ac} \right) \delta\theta_i^a \right) = 0. \tag{6}$$

After transforming the double sum for $C_a$ and $B$ of Eq. (6) into two separate sums for $I$ and $E$, and noting that the contact forces and the moments are opposite at internal contact (i.e., $f_i^c = f_i^{ac} = -f_i^{bc}$ and $m_i^c = m_i^{ac} = -m_i^{bc}$), one obtains the principle of virtual work:

$$\delta W_{\mathrm{I}}^D + \delta W_{\mathrm{E}}^D = 0, \tag{7}$$

where the work $\delta W_{\mathrm{E}}^D$ done by external forces and moments and the work $\delta W_{\mathrm{I}}^D$ done by internal forces and moments are

$$\delta W_{\mathrm{I}}^D = \sum_{c \in I} \left( f_i^c\, \Delta\delta u_i^c + m_i^c\, \Delta\delta\theta_i^c \right) \quad \text{and} \quad \delta W_{\mathrm{E}}^D = -\sum_{e \in E} \left( f_i^e \delta u_i^e + m_i^e \delta\theta_i^e \right). \tag{8}$$

As shown in Fig. 2, $\Delta\delta u_i^c$ and $\Delta\delta\theta_i^c$ are the relative displacement and rotation of the two particles $a$ and $b$ at their contact point $c$, respectively:

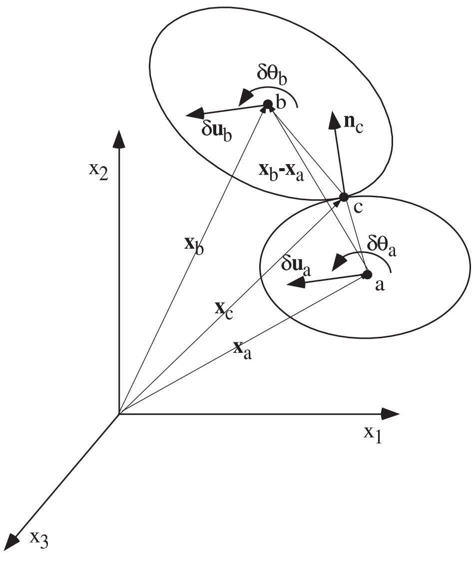


Fig. 2. Contact between two particles in a granular medium.

$$\Delta\delta u_i^c = \delta u_i^b - \delta u_i^a + e_{ijk}\Big(\delta\theta_j^b(x_k^c - x_k^b) - \delta\theta_j^a(x_k^c - x_k^a)\Big) \quad \text{and} \quad \Delta\delta\theta_i^c = \delta\theta_i^b - \delta\theta_i^a, \tag{9}$$

where $\delta u_i^e$ and $\delta\theta_i^e$ are the displacement and rotation of the points $e$ of application of external forces and moments. The variational displacements and rotations $\delta u_i^a$ and $\delta\theta_i^a$ can be selected arbitrarily. In particular, they can be chosen as follows:

$$\delta u_i^a = a_i + b_{ij}x_j^a + c_{ijk}x_j^a x_k^a \quad \text{and} \quad \delta\theta_i^a = \alpha_i + \beta_{ij}x_j^a \qquad i = 1,\ 2,\ 3, \tag{10}$$

where $a_i$, $b_{ij}$, $c_{ijk}$, $\alpha_i$, and $\beta_{ij}$ are arbitrary coefficients. By using Eq. (10), $\Delta\delta u_i^c$, $\Delta\delta\theta_i^c$ and $\delta u_i^e$ become

$$\Delta\delta u_i^c = b_{ij}(x_j^b - x_j^a) + c_{ijk}(x_j^b x_k^b - x_j^a x_k^a) - \alpha_j e_{ijk}\left(x_k^b - x_k^a\right) + \beta_{jl}e_{ijk}\left(x_l^b\left(x_k^c - x_k^b\right) - x_l^a\left(x_k^c - x_k^a\right)\right), \tag{11}$$

$$\Delta\delta\theta_i^c = \beta_{ij}(x_j^b - x_j^a), \tag{12}$$

$$\delta u_i^e = \delta u_i^a + e_{ijk}\delta\theta_j^a\left(x_k^e - x_k^{ae}\right) = a_i + b_{ij}x_j^{ae} + c_{ijk}x_j^{ae}x_k^{ae} + e_{ijk}\alpha_j\left(x_k^e - x_k^{ae}\right) + e_{ijk}\beta_{jl}x_l^{ae}\left(x_k^e - x_k^{ae}\right), \tag{13}$$

where $x^{ae}$ corresponds to the center of particle $a$, where contact $e$ takes place. By using Eqs. (11)–(13), $\delta W_{\mathrm{I}}^D$ and $\delta W_{\mathrm{E}}^D$ become

$$\begin{aligned}\delta W_{\mathrm{I}}^D = {} & b_{ij}\sum_{c\in I} f_i^c\left(x_j^b - x_j^a\right) + c_{ijk}\sum_{c\in I} f_i^c\left(x_j^b x_k^b - x_j^a x_k^a\right) - \alpha_j\sum_{c\in I} e_{ijk}f_i^c\left(x_k^b - x_k^a\right) \\ & + \beta_{jl}\sum_{c\in I}\Big(e_{ijk}f_i^c\left(x_l^b(x_k^c - x_k^b) - x_l^a(x_k^c - x_k^a)\right) + m_j^c(x_l^b - x_l^a)\Big),\end{aligned} \tag{14}$$

$$\begin{aligned}\delta W_{\mathrm{E}}^D = {} & -a_i\sum_{e\in E} f_i^e - b_{ij}\sum_{e\in E} f_i^e x_j^{ae} - c_{ijk}\sum_{e\in E} f_i^e x_j^{ae}x_k^{ae} - \alpha_j\sum_{e\in E}\Big(e_{ijk}f_i^e(x_k^e - x_k^{ae}) + m_j^e\Big) \\ & - \beta_{jl}\sum_{e\in E}\Big(e_{ijk}f_i^e(x_k^e - x_k^{ae}) + m_j^e\Big)x_l^{ae}.\end{aligned} \tag{15}$$

As Eqs. (7), (14) and (15) hold for arbitrary values of $a_i$, $b_{ij}$, $c_{ijk}$, $\alpha_i$ and $\beta_{ij}$, the following relations are obtained:

$$\sum_{e\in E} f_i^e = 0, \quad i = 1,\ 2,\ 3, \tag{16}$$

$$\sum_{c\in I}\left(x_i^b - x_i^a\right)f_j^c = \sum_{e\in E} x_i^{ae}f_j^e, \quad i,j = 1,\ 2,\ 3, \tag{17}$$

$$\sum_{c\in I} e_{ijk}(x_j^b - x_j^a)f_k^c = -\sum_{e\in E} M_i^{ae}, \quad i = 1,\ 2,\ 3, \tag{18}$$

$$\sum_{c\in I}\Big(e_{ikl}f_l^c\Big(x_j^b(x_k^c - x_k^b) - x_j^a(x_k^c - x_k^a)\Big) + m_i^c(x_j^b - x_j^a)\Big) = \sum_{e\in E} M_i^{ae}x_j^{ae}, \quad i,j = 1,\ 2,\ 3, \tag{19}$$

$$\sum_{c\in I} f_i^c(x_j^b x_k^b - x_j^a x_k^a) = \sum_{e\in E} f_i^{ae}x_j^{ae}x_k^{ae}, \quad i,j,k = 1,\ 2,\ 3, \tag{20}$$

where $M_i^{ae}$ is the external moment acting on particle $a$ about the center of particle $a$:

$$M_i^{ae} = e_{ijk}(x_j^e - x_j^{ae})f_k^e + m_i^e. \tag{21}$$

Eq. (16) translates the equilibrium of external forces applied to the whole assembly of particles. By using Eqs. (17) and (18), one can derive the equilibrium of external moments about the coordinate origin for the assembly of particles, i.e.,

$$\sum_{e\in E}\left(e_{imn}x_m^e f_n^e + m_i^e\right) = 0. \tag{22}$$

Therefore, Eq. (18) becomes

$$\sum_{c\in I} e_{ijk}(x_j^b - x_j^a)f_k^c = -\sum_{e\in E} M_i^{ae} = \sum_{e\in E} e_{imn}x_m^{ae} f_n^e, \quad i = 1,\ 2,\ 3. \tag{23}$$

For a volume $V$ to be in equilibrium, the sum of external moments about a common point must vanish (i.e., Eq. (22)). However, it is emphasized that the sum of external moments about different particle centers (i.e., $\sum_{e\in E} M_i^{ae}$) is not necessarily equal to zero. $M_i^{ae}$ results from not only contact moments $m_i^e$, but also contact forces $f_i^e$. It may be different from zero even where there is no contact moment.

## 3. Continuum for granular media

In the continuum equivalent for granular media, the traction vector $T_i$ and moment vector $m_i$ acting on the unit surface of unit normal vector $n_i$ is related to the Cauchy stress tensor $\sigma_{ij}$ and the couple stress tensor $\mu_{ij}$ through

$$T_i = \sigma_{ji}n_j \quad \text{and} \quad m_i = \mu_{ji}n_j, \qquad i = 1,\ 2,\ 3. \tag{24}$$

In the absence of external body force and moment per unit volume, the equations for equilibrium of internal stress and couple stress are

$$\sigma_{ji,j} = 0, \quad i = 1,\ 2,\ 3, \tag{25}$$

$$\mu_{ji,j} + e_{ikl}\sigma_{kl} = 0, \quad i = 1,\ 2,\ 3. \tag{26}$$

The kinematics of the equivalent continuum is defined by the fields of displacement vector $\delta u_i$ and rotation $\delta\theta_i$, which describe the motion of particle centers, i.e.,

$$\delta u_i(x_j^a) = \delta u_i^a \quad \text{and} \quad \delta\theta_i(x_j^a) = \delta\theta_i^a \qquad \forall a \in B. \tag{27}$$

By multiplying Eqs. (25) and (26) by any variational fields $\delta u_i$ and $\delta\theta_i$, and integrating over volume $V$, one obtains the following relation:

$$\int_V \left(\sigma_{ji,j}\ \delta u_i + (\mu_{ji,j} + e_{ikl}\sigma_{kl})\ \delta\theta_i\right) = 0. \tag{28}$$

By invoking the Gauss theorem, the principle of virtual work is obtained:

$$\delta W_\mathrm{I} + \delta W_\mathrm{E} = 0, \tag{29}$$

where the virtual work $\delta W_\mathrm{E}$ of external forces and moments and the virtual work $\delta W_\mathrm{I}$ of internal stresses are

$$\delta W_\mathrm{I} = \int_V \left(\sigma_{ji}(\delta u_{i,j} + e_{ijk}\,\delta\theta_k) + \mu_{ji}\,\delta\theta_{i,j}\right)\mathrm{d}V \quad \text{and} \quad \delta W_\mathrm{E} = -\int_S (T_i\,\delta u_i + m_i\,\delta\theta_i)\,\mathrm{d}S. \tag{30}$$

By choosing $\delta u_i$ and $\delta\theta_i$ as specified in Eq. (10), Eq. (30) becomes

$$\delta W_{\mathrm{E}} = -a_i \int_S T_i \,\mathrm{d}S - b_{ij} \int_S T_i x_j \,\mathrm{d}S - c_{ijk} \int_S T_i x_j x_k \,\mathrm{d}S - \alpha_i \int_S m_i \,\mathrm{d}S - \beta_{ij} \int_S m_i x_j \,\mathrm{d}S, \tag{31}$$

$$\delta W_{\mathrm{I}} = b_{ij} \int_V \sigma_{ji} \,\mathrm{d}V + c_{ijk} \int_V (\sigma_{ji} x_k + \sigma_{ki} x_j) \,\mathrm{d}V - \alpha_i e_{ijk} \int_V \sigma_{jk} \,\mathrm{d}V + \beta_{ij} \int_V \left(\mu_{ji} + e_{ikl} x_j \sigma_{lk}\right) \mathrm{d}V. \tag{32}$$

As Eqs. (29), (31) and (32) hold for any values of $a_i$, $b_{ij}$, $c_{ijk}$, $\alpha_i$, and $\beta_{ij}$, the following relations are obtained:

$$\int_S T_i \,\mathrm{d}S = 0, \quad i = 1,\ 2,\ 3, \tag{33}$$

$$\int_V \sigma_{ij} \,\mathrm{d}V = \int_S x_i T_j \,\mathrm{d}S, \quad i,j = 1,\ 2,\ 3, \tag{34}$$

$$e_{ijk} \int_V \sigma_{jk} \,\mathrm{d}V = -\int_S m_i \,\mathrm{d}S, \quad i = 1,\ 2,\ 3, \tag{35}$$

$$\int_V \left(\mu_{ij} + e_{jkl} x_i \sigma_{lk}\right) \mathrm{d}V = \int_S x_i m_j \,\mathrm{d}S, \quad i,j = 1,\ 2,\ 3, \tag{36}$$

$$\int_V (\sigma_{ji} x_k + \sigma_{ki} x_j) \,\mathrm{d}V = \int_S T_i x_j x_k \,\mathrm{d}S, \quad i,j = 1,\ 2,\ 3. \tag{37}$$

Eq. (33) implies the equilibrium of external forces. Eq. (34) represents the average stress $\bar{\sigma}_{ij}$ in volume $V$:

$$\bar{\sigma}_{ij} = \frac{1}{V} \int_V \sigma_{ij} \,\mathrm{d}V = \frac{1}{V} \int_S x_i T_j \,\mathrm{d}S. \tag{38}$$

Eq. (35) is useful to examine the symmetry of $\bar{\sigma}_{ij}$:

$$e_{ikj} \bar{\sigma}_{kj} = -\frac{1}{V} \int_S m_i \,\mathrm{d}S, \quad i = 1,\ 2,\ 3, \tag{39}$$

where $e_{ikj}\bar{\sigma}_{kj} = 0$ when the stress tensor is symmetric (i.e., $\bar{\sigma}_{ij} = \bar{\sigma}_{ji}$). However, $\int_S m_i \,\mathrm{d}S$ is not necessarily equal to zero when there are moments at the external boundary. As previously mentioned, these external moments in granular media may result from contact forces without contact moments. Finally, Eq. (36) becomes

$$\frac{1}{V} \int_V \left(\mu_{ij} + e_{jkl} x_i \sigma_{lk}\right) \mathrm{d}V = \bar{\mu}_{ij} + e_{jkl} \bar{\Sigma}_{ilk}, \tag{40}$$

where $\bar{\Sigma}_{ijk}$ is the average moment of stress:

$$\bar{\Sigma}_{ijk} = \frac{1}{V} \int_V x_i \sigma_{jk} \,\mathrm{d}V. \tag{41}$$

In summary, the internal work becomes

$$\delta W_{\mathrm{I}} = V\Big(b_{ij}\bar{\sigma}_{ji} + c_{ijk}(\bar{\Sigma}_{kji} + \bar{\Sigma}_{jki}) - \alpha_i e_{ijk} \bar{\sigma}_{jk} + \beta_{ij}(\bar{\mu}_{ji} + e_{ikl}\bar{\Sigma}_{jlk})\Big). \tag{42}$$

## 4. Definition of average stresses in granular media

The average stresses in the equivalent continuum are defined by postulating that the granular and continuous media produce identical internal and external works:

$$\delta W_{\mathrm{I}}^{D} = \delta W_{\mathrm{I}} \quad \text{and} \quad \delta W_{\mathrm{E}}^{D} = \delta W_{\mathrm{E}}. \tag{43}$$

### 4.1. Average stress

As Eq. (43) applies to arbitrary values of $b_{ij}$, the average stress is

$$\bar{\sigma}_{ij} = \frac{1}{V}\sum_{c\in I}(x_i^b - x_i^a)f_j^c = \frac{1}{V}\sum_{e\in E}x_i^{ae}f_j^e. \tag{44}$$

Eq. (44) is identical to those derived by Weber (1966), Goddard (1977), Christoffersen et al. (1981), and Rothenberg and Selvadurai (1981). In the case of spherical and cylindrical particles $a$ and $b$, which are in contact at point $c$ with unit normal vector $n_i^c$ (i.e., $x_i^b - x_i^a = (R_a + R_b)n_i^c$), due to the opposite sign of contact forces and contact normals (i.e., $f_i^{ca} = -f_i^{cb}$ and $n_i^{ca} = -n_i^{cb}$), Eq. (44) becomes

$$\bar{\sigma}_{ij} = \frac{1}{V}\sum_{c\in I}(R_a + R_b)n_i^c f_j^c = \frac{1}{V}\sum_{a\in B}\sum_{c\in I_a}R_a n_i^{ca} f_j^{ca}. \tag{45}$$

### 4.2. Symmetry of average stress

As Eq. (43) applies to arbitrary values of $\alpha_j$, one obtains

$$e_{ijk}\bar{\sigma}_{jk} = \frac{1}{V}\sum_{c\in I}e_{ijk}(x_j^b - x_j^a)f_k^c = -\frac{1}{V}\sum_{e\in E}M_i^{ae} = \frac{1}{V}\sum_{e\in E}e_{ijk}x_j^{ae}f_k^e, \quad i = 1,\ 2,\ 3. \tag{46}$$

Eq. (46), which can also be obtained directly from Eq. (44), is useful to determine the amplitude of stress asymmetry. This amplitude can also be characterized by $\bar{\sigma}_{ij} - \bar{\sigma}_{ji}$ as follows:

$$\bar{\sigma}_{ij} - \bar{\sigma}_{ji} = \frac{1}{V}\sum_{e\in E}\left(x_i^{ae}f_j^e - x_j^{ae}f_i^e\right) = -(e_{ijk} - e_{jik})\frac{1}{V}\sum_{e\in E}M_k^{ae}. \tag{47}$$

Eq. (47) implies that the average stress may be asymmetric, even when there is no moment at contacts (i.e., $m_i^e = 0$). The asymmetry results from the sum of the external moments that are created by external forces $f_i^e$ about the particle centers.

The amplitude of stress asymmetry increases with the area $S$ on which the external moments are applied, but decreases with volume $V$. If the external moments are assumed to have bounded values, the amplitude of stress asymmetry decreases with $V/S$. When $V \to \infty$, the effects of external moments vanish, and the average stress is symmetric. This applies with or without contact moments.

In the case of spherical and cylindrical particles, Eq. (47) becomes

$$\bar{\sigma}_{ij} - \bar{\sigma}_{ji} = \frac{1}{V}\sum_{a\in B}\sum_{c\in I_a}R_a\left(n_i^{ac}f_j^{ac} - n_j^{ac}f_i^{ac}\right) = \frac{1}{V}\sum_{c\in I}(R_a + R_b)\left(n_i^{ac}f_j^{ac} - n_j^{ac}f_i^{ac}\right). \tag{48}$$

Eq. (48) shows that $\bar{\sigma}_{ij}$ is not necessarily symmetric when the particles are spheres or cylinders of identical radius. This result, which is in disagreement with Caillerie (1991) and Chang and Liao (1990), will later be verified in a particular example.

### 4.3. Average micropolar stress and first moment of stress

As Eq. (43) applies to arbitrary values of $\beta_{ij}$ and $c_{ijk}$, the following relations are obtained:

$$\bar{\mu}_{ji} + e_{ikl}\bar{\Sigma}_{jlk} = \frac{1}{V}\sum_{c\in I}\Big(e_{ikl}f_l^c\Big(x_j^b(x_k^c - x_k^b) - x_j^a(x_k^c - x_k^a)\Big) + m_i^c(x_j^b - x_j^a)\Big)$$
$$= \frac{1}{V}\sum_{e\in E}M_i^{ae}x_j^{ae}, \qquad i,j = 1,\ 2,\ 3, \tag{49}$$

$$\bar{\Sigma}_{kji} + \bar{\Sigma}_{jki} = \frac{1}{V}\sum_{c\in I}f_i^c(x_j^b x_k^b - x_j^a x_k^a) = \frac{1}{V}\sum_{e\in E}f_i^{ae}x_j^{ae}x_k^{ae}, \quad i,j,k = 1,2,3. \tag{50}$$

Therefore, the external moments $M_i^{ae}$, which result from external contact force $f_i^e$ and/or contact moment $m_i^e$, generate not only asymmetric stress, but also couple stress and first stress moment. This result is in agreement with Eq. (26), which states that couple stresses are required to balance asymmetric stresses. However, the present approach provides only the sums $\bar{\mu}_{ji} + e_{ikl}\bar{\Sigma}_{jlk}$ and $\bar{\Sigma}_{kji} + \bar{\Sigma}_{jki}$, and unfortunately not each term $\bar{\mu}_{ji}$ and $\bar{\Sigma}_{jlk}$.

## 5. Alternate definition of average stress

The average stress can also be defined based on statics, instead of virtual work (Cundall and Strack, 1978–1979). The average stress within volume $V$ is defined as the weighted average of the stress $\bar{\sigma}_{ij}^a$ for each particle $a$ of $B$:

$$\bar{\sigma}_{ij}^* = \frac{1}{V}\sum_{a\in B}\bar{\sigma}_{ij}^a V_a, \tag{51}$$

where $V_a$ is the volume of particle $a$, and

$$\bar{\sigma}_{ij}^a = \frac{1}{V_a}\int_{V_a}\sigma_{ij}\,\mathrm{d}V. \tag{52}$$

By using Eq. (34), and replacing the traction vector $T_i$ with discrete contact force $f_i^c$, $\bar{\sigma}_{ij}^a$ becomes

$$\bar{\sigma}_{ij}^a = \frac{1}{V_a}\int_{S_a}x_i T_j\,\mathrm{d}S = \frac{1}{V_a}\sum_{c\in C_a}x_i^c f_j^c. \tag{53}$$

Because the contact forces are opposite at internal contacts (i. e., $f_j^{ca} = -f_j^{cb}$), the average stress $\bar{\sigma}_{ij}^*$ is

$$\bar{\sigma}_{ij}^* = \frac{1}{V}\sum_{a\in B}\sum_{c\in C_a}x_i^c f_j^c = \frac{1}{V}\sum_{c\in I}x_i^c(f_j^{ca} + f_j^{cb}) + \frac{1}{V}\sum_{e\in E}x_i^e f_j^e = \frac{1}{V}\sum_{e\in E}x_i^e f_j^e. \tag{54}$$

Note that $x_i^e$ in Eq. (54) refers to contact point $e$, whereas $x_i^{ae}$ in Eq. (44) refers to the center of the particle $a$, where contact $e$ takes place. $\bar{\sigma}_{ij}^*$ and $\bar{\sigma}_{ij}$ are related through

$$\bar{\sigma}_{ij}^* = \bar{\sigma}_{ij} + \frac{1}{V}\sum_{e\in E}(x_i^e - x_i^{ae})f_j^e. \tag{55}$$

The symmetry of $\bar{\sigma}_{ij}^a$ results from the equilibrium of moments about the coordinate origins for particle $a$, i.e.,

$$e_{ijk}\bar{\sigma}^a_{jk} = \frac{1}{V_a}\sum_{c\in C_a} e_{ijk}x^c_j f^c_k = 0, \quad i = 1,\ 2,\ 3. \tag{56}$$

The stress $\bar{\sigma}^*_{ij}$ is therefore symmetric, because it is the weighted sum of symmetric $\bar{\sigma}^a_{ij}$. The symmetry of $\bar{\sigma}^*_{ij}$ can also be shown by using Eq. (54) and invoking the equilibrium of external moments about the coordinate origin (i.e., Eq. (22)).

The symmetry of $\bar{\sigma}^*_{ij}$ has significant implications in computational granular mechanics, especially for the computer simulations using dynamic relaxation to solve the equilibrium equations of statics (Cundall and Strack, 1978–1979; Bardet and Proubet, 1991). When there is no moment at contacts, $\bar{\sigma}^*_{ij}$ should be symmetric, and any computed asymmetry of $\bar{\sigma}^*_{ij}$ should be interpreted as inaccurate calculation and/or lack of static equilibrium.

In the case of spherical and cylindrical particles of radius $R_a$, $x^c_i = x^a_i + R_a n^c_i$, the average stress $\bar{\sigma}^*_{ij}$ becomes

$$\bar{\sigma}^*_{ij} = \frac{1}{V}\sum_{a\in B}\frac{1}{V_a}\sum_{c\in C_a}(x^a_i + R_a n^c_i)f^c_j = \frac{1}{V}\sum_{a\in B}\frac{R_a}{V_a}\sum_{c\in C_a} n^c_i f^c_j = \frac{1}{V}\sum_{a\in B} R_a \sum_{c\in C_a} n^c_i f^c_j. \tag{57}$$

Eq. (57) is the same as that obtained by Cundall and Strack (1979).

## 6. Examples

We will illustrate the circumstances of stress asymmetry in the case of double and multiple layer interfaces filled with cylindrical or spherical particles.

### *6.1. Double layer interface*

The stresses $\bar{\sigma}_{ij}$ and $\bar{\sigma}^*_{ij}$ can be calculated analytically for the particular example of Fig. 3, which represents an interface made of $p$ columns each having two particles. The columns have the same height, but are made of particles of various diameters. The particle assembly is subjected to the normal force $N_p$ and shear force $S_p$, which are assumed to be distributed evenly onto each particle column; the normal and shear forces acting at all contacts are $N = N_p/p$ and $S = S_p/p$, respectively. The contact direction between two particles is identical for all particles in contact. It is characterized by the unit vector **n** of component

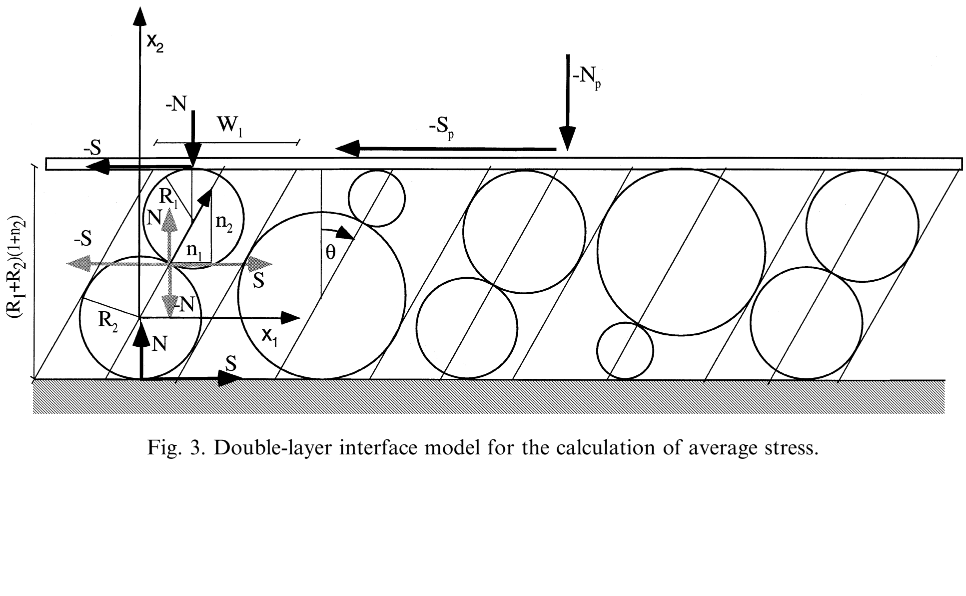


Fig. 3. Double-layer interface model for the calculation of average stress.

$n_1 = \sin\theta$ and $n_2 = \cos\theta$. The equilibrium of forces and moments for all particles and the top plate are satisfied when

$$S = Nn_1/(1+n_2) = N\tan(\theta/2). \tag{58}$$

All the contact forces have the same inclination $\theta/2$ relative to the contact direction **n**. The contacts do not slip and the interface remains stable as long as $\theta$ remains smaller than $2\phi$, where $\phi$ is the friction angle between two particles. For the calculation of $\bar{\sigma}_{ij}$, it is convenient to select the coordinate axis at the center of particle 2. In this coordinate system, the center coordinates of particle 1 are

$$x_1^{1e} = n_1 L \quad \text{and} \quad x_2^{1e} = n_2 L, \tag{59}$$

where $L = R_1 + R_2$. By using Eqs. (44) and (59), the stresses $\bar{\sigma}_{ij}$ are

$$\bar{\sigma}_{11} = -n_1 S/A, \quad \bar{\sigma}_{22} = -\frac{(1+n_2)n_2}{n_1} S/A, \quad \bar{\sigma}_{12} = -(1+n_2)S/A, \quad \text{and} \quad \bar{\sigma}_{21} = -n_2 S/A, \tag{60}$$

where $A = V/L$, and $V$ is the average volume of particle columns. For interfaces filled with cylindrical or spherical particles, $V$ can be evaluated as follows:

$$V = (1+n_2)LW_1W_2, \tag{61}$$

where $W_1$ is the average width of particle columns in the $x_1$ direction. For cylindrical particles, $W_2$ is the particle length. For spherical particles, $W_2$ is the average width of particle columns in the $x_2$ direction. Both $W_1$ and $W_2$ depend on the density of particles in the interface.

The external moments $M_i^{ae}$ acting on particles 1 and 2 are

$$M_3^1 = R_1 S \quad \text{and} \quad M_3^2 = R_2 S. \tag{62}$$

By using Eq. (47) or Eq. (60), the amplitude of stress asymmetry is

$$\bar{\sigma}_{12} - \bar{\sigma}_{21} = -\frac{M_3^1 + M_3^2}{V} = -S/A. \tag{63}$$

The stress asymmetry vanishes when $S = 0$, i.e., when the particle columns are vertical. As previously stated in Eq. (48), $\bar{\sigma}_{ij}$ is not necessarily symmetric for particles of identical radius (e.g., $R_1 = R_2$). The couple stress and first stress moment is

$$\bar{\mu}_{13} + \bar{\Sigma}_{121} - \bar{\Sigma}_{112} = R_1 n_1 S/A \quad \text{and} \quad \bar{\mu}_{23} + \bar{\Sigma}_{221} - \bar{\Sigma}_{212} = R_1 n_2 S/A. \tag{64}$$

For the calculation of $\bar{\sigma}^*_{ij}$, it is convenient to select the coordinate axis at the lowest external contact. The coordinates of the highest contact are therefore

$$x_1^e = n_1 L \quad \text{and} \quad x_2^e = (1+n_2)L, \tag{65}$$

and the stress $\bar{\sigma}^*_{ij}$ is

$$\bar{\sigma}^*_{11} = -n_1 S/A, \quad \bar{\sigma}^*_{22} = -\frac{(1+n_2)^2}{n_1} S/A, \quad \text{and} \quad \bar{\sigma}^*_{12} = \bar{\sigma}^*_{21} = -(1+n_2)S/A. \tag{66}$$

As expected, $\bar{\sigma}^*_{ij}$ is symmetric. $\bar{\sigma}^*_{ij}$ and $\bar{\sigma}_{ij}$ are related through

$$\bar{\sigma}^*_{11} = \bar{\sigma}_{11}, \quad \bar{\sigma}^*_{22} = \bar{\sigma}_{22} - \frac{(1+n_2)}{n_1} S/A, \quad \bar{\sigma}^*_{12} = \bar{\sigma}_{21}, \quad \text{and} \quad \bar{\sigma}^*_{21} = \bar{\sigma}_{21} - S/A. \tag{67}$$

Fig. 4 compares the normalized variation of $\bar{\sigma}^*_{ij}$ and $\bar{\sigma}_{ij}$ when the angle $\theta = \tan^{-1}(n_1/n_2)$ varies from 0° to 90°. The stress asymmetry (i.e., $\bar{\sigma}_{12} - \bar{\sigma}_{21}$) remains constant and independent from $\theta$.

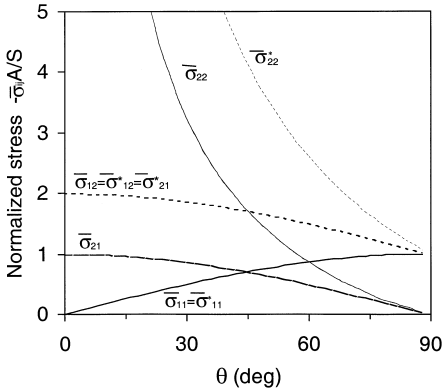


Fig. 4. Variation of normalized stresses of $\bar{\sigma}_{ij}A/S$ and $\bar{\sigma}^*_{ij}A/S$ with angle $\theta$.

## 6.2. Multi-layer interface

As shown in Fig. 5, the multi-layer interface model is made of $p$ columns of particles, each column $i$ having $q_i$ particles ($q_i$ must be an even number). This multi-layer model is a generalization of the two-layer model, which corresponds to $q_i = 2$ for $i = 1, \ldots, p$. The particle columns have the same height and contact direction $\mathbf{n}$, but may have different numbers $q_i$ of particles. All the particles and the top platen are in static equilibrium, when Eq. (58) is satisfied; the contact forces are identical to those of the two-layer model. The

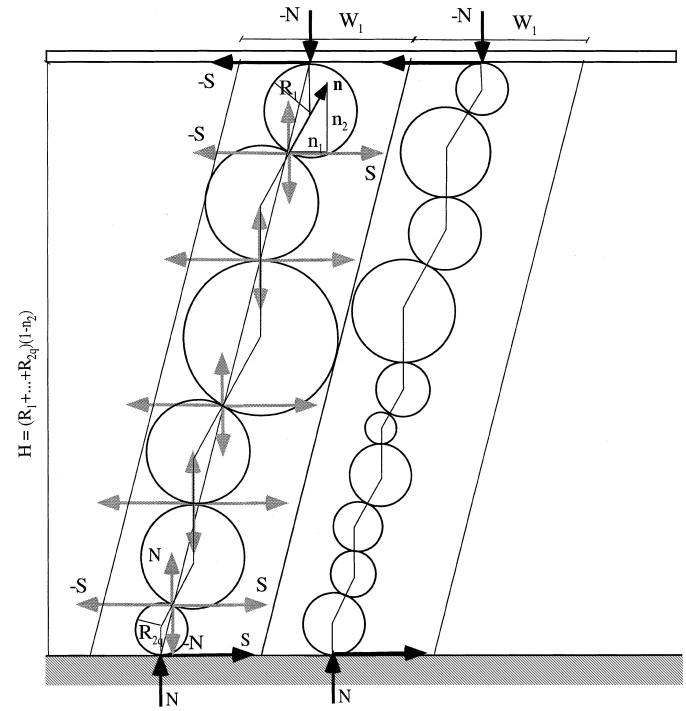


Fig. 5. Multi-layer interface model for the calculation of average stress.

average stresses in the interface can be determined by averaging the average stresses in each column. Hereafter, we will calculate only the average stresses in a column.

After selecting the coordinate axis at the center of particle $q_i$, the center coordinates of particle 1 are

$$x_1 = n_1 L \quad \text{and} \quad x_2 = (1 + n_2)L - (R_1 + R_{q_i}), \tag{68}$$

where $L = \sum_{j=1}^{q_i} R_j$. The average stresses $\bar{\sigma}_{ij}$ in column $i$ are

$$\begin{aligned} &\bar{\sigma}_{11} = -n_1 S/A, \quad \bar{\sigma}_{22} = -\frac{(1+n_2)^2}{n_1} S/A + \frac{R_1 + R_{q_i}}{V} \frac{1+n_2}{n_1} S, \quad \bar{\sigma}_{12} = -(1+n_2)S/A, \quad \text{and} \\ &\bar{\sigma}_{21} = -(1+n_2)S/A + \frac{R_1 + R_{q_i}}{V} S, \end{aligned} \tag{69}$$

where $A$ and $V$ are defined as for the double layer interface. The values of $\bar{\sigma}_{11}$ and $\bar{\sigma}_{12}$ are identical to those in Eq. (60), whereas the corresponding values of $\bar{\sigma}_{22}$ and $\bar{\sigma}_{21}$ are different. The differences, however, vanish when $V \to \infty$. The stresses $\bar{\sigma}^*_{ij}$ in the multi-layer are identical to those of the double layer interface (i.e., Eq. (66)). As predicted by Eq. (69), $\bar{\sigma}_{ij}$ converges toward $\bar{\sigma}^*_{ij}$ when volume $V$ becomes large (i.e., $q \to \infty$). The external moments $M_i^{ae}$ acting on particle 1 and $q_i$ are

$$M_3^1 = R_1 S \quad \text{and} \quad M_3^{q_i} = R_{q_i} S. \tag{70}$$

These external moments are responsible for the stress asymmetry as follows:

$$\bar{\sigma}_{12} - \bar{\sigma}_{21} = -S(R_1 + R_{q_i})/V. \tag{71}$$

The amplitude of stress asymmetry decreases with the number of particles in the columns, and increases with the size of particles at the top and bottom of columns. The stress becomes symmetric when the column height becomes infinite (i.e., $q \to \infty$). The couple stress and first stress moment are

$$\begin{aligned} &\bar{\mu}_{13} + \bar{\Sigma}_{121} - \bar{\Sigma}_{112} = R_1 n_1 S/A, \\ &\bar{\mu}_{23} + \bar{\Sigma}_{221} - \bar{\Sigma}_{212} = R_1(1+n_2)S/A - R_1(R_1 + R_{q_i})S/V. \end{aligned} \tag{72}$$

In contrast to the asymmetric stress components, $\bar{\mu}_{ij} + e_{ikl}\bar{\Sigma}_{jlk}$ does not decrease when volume $V$ becomes large (i.e., $q \to \infty$).

## 7. Discussion

The asymmetry of stress depends on the way stresses are defined. The stresses defined by statics are symmetric when there is no contact moment. However, the stresses defined by virtual work may become asymmetric when there is no contact moment. Our analysis differs from the previous ones (Christoffersen et al., 1981; Chang and Liao, 1990), because we considered external moments, and established the circumstances and amplitude of stress asymmetry. In agreement with Caillerie (1991), we found that the asymmetry originates from external moments, and that the amplitude of stress asymmetry decreases with the size of the granular volume. The stress asymmetry is, therefore, more detectable in elongated samples subjected to external moments on their boundary. Bulky samples subjected to small external moments are likely to display negligible stress asymmetry. The stress asymmetry can rightfully be neglected in large masses of granular media far away from the boundaries with external moments. However, it may become important in interfaces with significant external moments. There is a need for verifying these findings through computer simulations and laboratory experiments.

## 7.1. Remark on volume averaging

The areas for calculating average stresses can be selected in various ways. In two dimensions, these areas are usually chosen to be circular. In this case, the mean normal traction $\langle t_n \rangle$ becomes equal to the trace of the stress tensor $\sigma_{ij}$ (Vardoulakis and Sulem, 1995), i.e.,

$$\langle t_n \rangle = \frac{1}{S} \int \int t_i n_i \, \mathrm{d}S = \frac{1}{S} \int \int \sigma_{ij} n_i n_j \, \mathrm{d}S = \sigma_{kk}, \tag{73}$$

where $S$ is the circle perimeter, and $n_i$ the unit normal to $S$. As pointed out by Novozhilov (1961), the value of $\langle t_n \rangle$ strongly depends on the shape of the averaging area. For example, in the case of the rectangular area of Fig. 6, the normal $\langle t_n \rangle$and tangential $\langle t_i \rangle$ components of the mean traction are

$$\langle t_n \rangle = \frac{l_2}{l_1 + l_2} \sigma_{11} + \frac{l_1}{l_1 + l_2} \sigma_{22} \quad \text{and} \quad \langle t_t \rangle = \frac{1}{l_1 + l_2} \sqrt{(l_2 \sigma_{12})^2 + (l_1 \sigma_{21})^2}, \tag{74}$$

where $l_1$ and $l_2$ characterize the rectangular dimensions. The area shape can be selected to emphasize the structural anisotropy of the granular assembly over which the average is calculated. For instance, Eq. (74) becomes

$$\langle t_n \rangle = \frac{\alpha}{1 + \alpha} \sigma_{11} + \frac{1}{1 + \alpha} \sigma_{22} \quad \text{and} \quad \langle t_t \rangle = \sqrt{\left( \frac{\alpha}{1 + \alpha} \sigma_{12} \right)^2 + \left( \frac{1}{1 + \alpha} \sigma_{21} \right)^2} \tag{75}$$

after introducing the aspect ratio $\alpha = l_2 / l_1$. In the limit of a very elongated area (i.e., $\alpha \to 0$), Eq. (75) reduces to

$$\langle t_n \rangle \to \sigma_{22} \quad \text{and} \quad \langle t_t \rangle \to |\sigma_{21}|. \tag{76}$$

This implies that for *shear bands*, which are typically elongated structures, the dominant stress quantities are the shear and normal stresses acting on planes parallel to the bands. As shown in Fig. 6, when the normal stresses $S_{11}$ and $S_{22}$ vary linearly on the rectangle sides, i.e.,

$$S_{11} = \sigma_{11} + C_1 x_2 \quad \text{and} \quad S_{22} = \sigma_{22} + C_2 x_1, \tag{77}$$

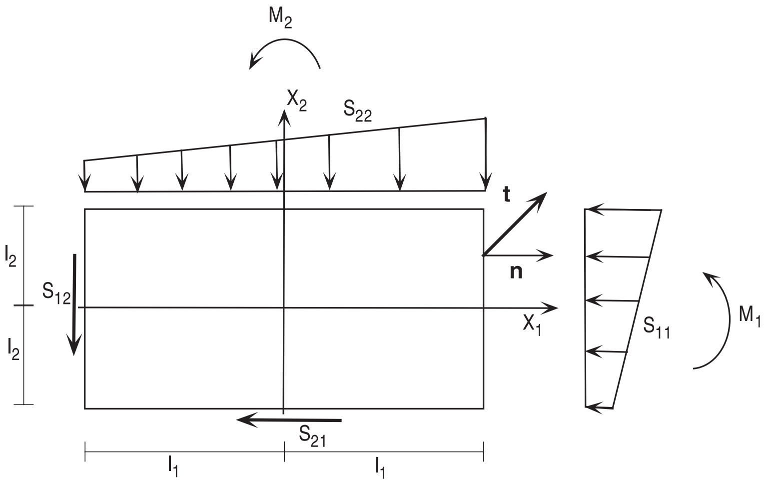


Fig. 6. Rectangular area for calculating the average stress.

where $C_1$ and $C_2$ are constant coefficients, the following couple stresses are obtained (Vardoulakis and Sulem, 1995):

$$M_1 = \tfrac{2}{3}C_1 l_2^3 \quad \text{and} \quad M_2 = \tfrac{2}{3}C_2 l_1^3. \tag{78}$$

Under these conditions, the moment equilibrium for the averaging area yields

$$S_{21} - S_{12} = \sigma_{21} - \sigma_{12} = 2\left(\alpha m_1 + \frac{1}{\alpha} m_2\right), \qquad m_1 = \frac{M_1}{2l_2} \quad \text{and} \quad m_2 = \frac{M_2}{2l_1}. \tag{79}$$

For the very elongated area (i.e., $\alpha \to 0$), the asymmetry of stress becomes

$$\sigma_{21} - \sigma_{12} \approx 2\frac{m_2}{\alpha} = \frac{M_2}{l_2}. \tag{80}$$

Eq. (80) implies that the couple stresses may result from the spatial variation of normal stresses acting on the planes parallel to the longer side of elongated sampling areas, and that the amplitude of these couple stresses scales with the smaller dimension of areas (e.g., $l_2$). In other words, the asymmetry of stress and the existence of couple stresses may result from the averaging procedures, which accounts for elongated structures like shear bands. The effects of averaging procedures on the asymmetry of average stress needs to be investigated further through computer simulations of discrete particle assemblies.

## 8. Conclusion

We have derived the conditions for the asymmetry of stress in granular materials, and shown that there is asymmetry even when the particle contacts do not transmit moments. This stress asymmetry is obtained when the stress is defined from virtual work, but is lost when the stress is defined from statics. The asymmetry results from external moments are applied by external forces. We also show that the amplitude of stress asymmetry decreases with the ratio $V/S$ between surface $S$ and volume $V$. When $V/S$ becomes very large, the stress asymmetry disappears, with or without external contact moments.

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